165 Math & Computers

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M,W,F 4:10 - 5:00 PM

Hart Hall 1150

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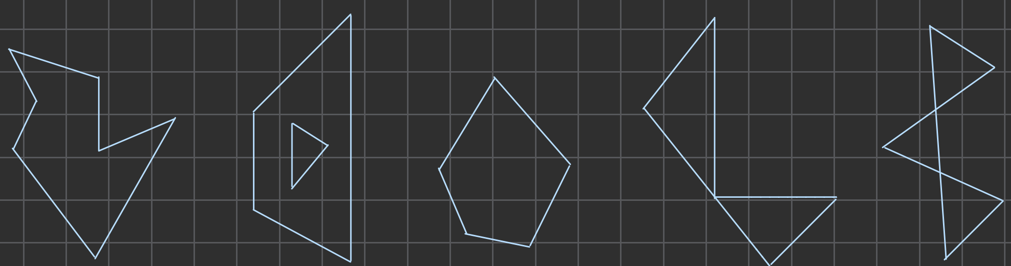
T,T 11:00 - 12:00

Polygons

Computational geometry is fundamentally discrete. Computation with curves and smooth surfaces are generally considered part of another field, often called "geometric modeling".

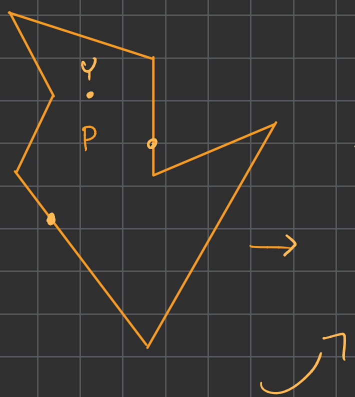
A polygon P is the closed region of the plane bounded by a finite collection of line segments forming a closed curve that does not intersect itself.



(a) (b) (c) (d) (e)

Theorem (Polygonal Jordan Curve). The boundary ∂P of a polygon P partitions the plane into two parts. In particular, the two components of $\mathbb{R}^2/\partial P$ are the bounded interior and the unbounded exterior.

Sketch of the proof.



Choose a fixed direction that is not parallel to any edge of P.

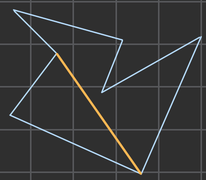
Then any point $x \in \mathbb{R}^2 \setminus \{\partial P\}$ lies in one of the following two sets:

A 1) The ray through x in direction u crosses ∂P in an even number of times.

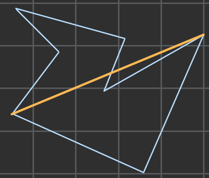
B 2) The ray through x in direction u crosses ∂P in an odd number of times.

Exercise. Prove that a)- Every path between points lying in different sets must cross ∂P.
b)- There is a path between points in the same set that doesn't contains points of ∂P.

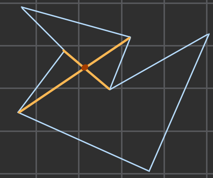
A diagonal of a polygon P is a line segment connecting two vertices of P and lying in the interior of P, not touching ∂P except at its endpoints.



(a)

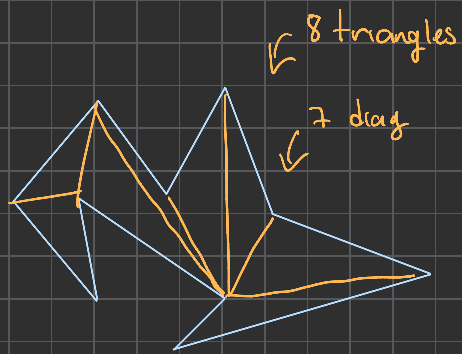


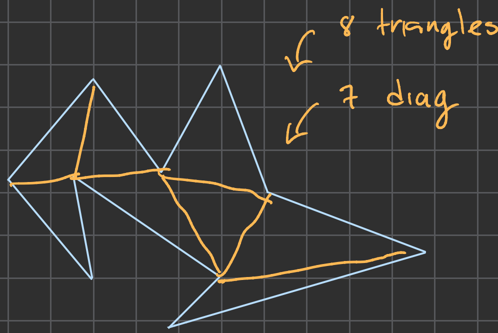
(b)



(c)

Definition. A triangulation of a polygon P is a decomposition of P into triangles by a maximal set of non crossing diagonals.

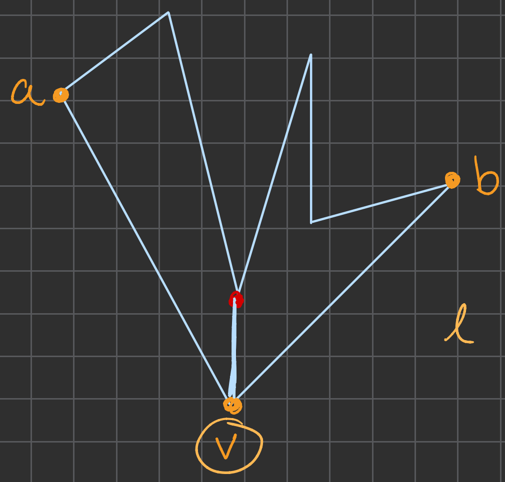




Some questions:

- How many different triangulations does a given polygon have?
- How many triangles are in each triangulation of a given polygon?
- Must every polygon always have at least one diagonal?

Lemma: Every polygon P with more 3 vertices has a diagonal.



Proof: Let be v the lowest vertex of P; If there are several take the rightmost. Let a and b the neighbors of v.

If the segment $\overline{ab}$ is contained in P and $\overline{ab} \cap \partial P = \{a, b\}$ then $\overline{ab}$ is a diagonal.

Otherwise, since P has more than three vertices, the closed triangle Δabv contains at least one vertex of P.

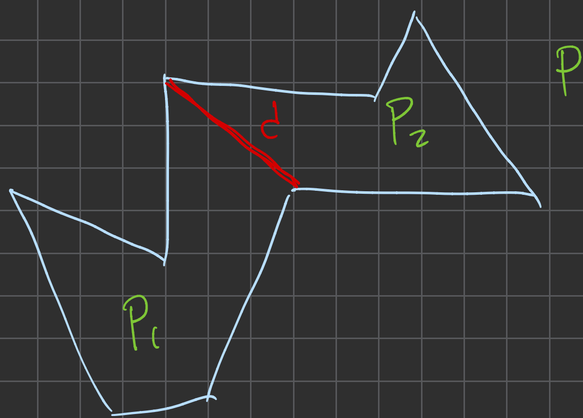
Let $\perp$ be a line parallel to $\overline{ab}$ through v. Sweep this line parallel to itself upward toward $\overline{ab}$.

Let x the first vertex different to a, b or v. The (shaded) triangular region of the polygon below line $\perp$ and above v is empty of vertices.

Because vx cannot int ∂P except at v and x vx is our diagonal. ∎

Theorem: Every polygon has a triangulation.

Proof:
- If P have 3 vertices ✓
- Suppose $|V| > 3$ and the thm is valid for polygons with fewer vertices

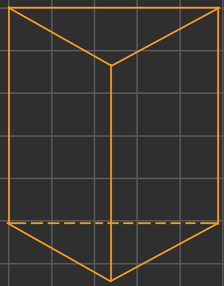


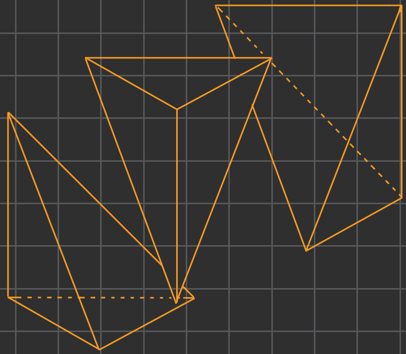
- There is a diag d s.t. d divides P into P_1 and P_2

Tetrahedralization

For a 3-dimensional polytope (polyhedron) P, we can "triangulate" P using tetrahedrons.

¿How many tetrahedrons?





Can all polyhedra be tetrahedralized?

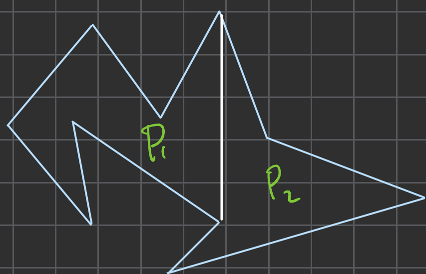
Open problem:

Find a characterization for tetrahedralizable polyhedra.

In 1992 Jim Ruppert and Raimund Seidel proved that determining whether a polyhedron is tetrahedralizable is NP-complete.

Theorem. Every triangulation of a polygon P with n vertices has $n-2$ triangles and $n-3$ diagonals.

Proof:



$|V(P_1)| = n_1$ $\qquad$ $|V(P_2)| = n_2$

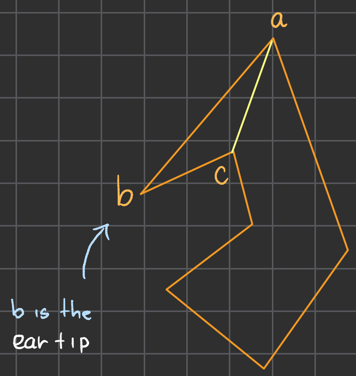
P_1 have
$n_1 - 2$ triang
$n_1 - 3$ edges

P_2 have
$n_2 - 2$ trian
$n_2 - 3$ diag

$$n_1 - 2 + n_2 - 2 = (n_1 + n_2) - 4$$
$$= n + 2 - 4$$
$$= n - 2$$

$$n_1 - 3 + n_2 - 3 + 1 =$$
$$n_1 + n_2 - 5 = n + 2 - 5 = n - 3$$

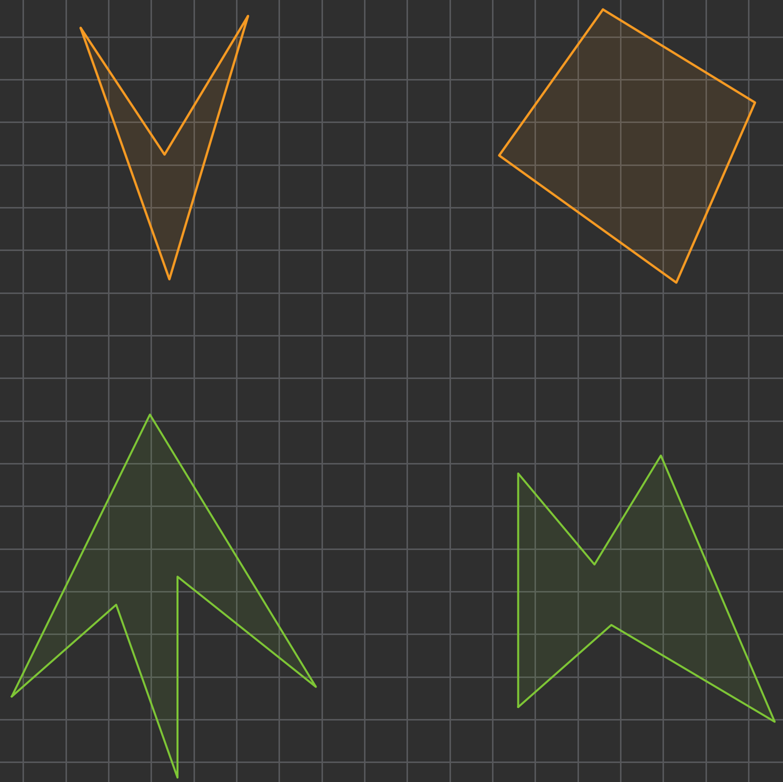
We sometimes call $\boxed{\text{ears}}$ three consecutive vertices a, b, c if ac is a diagonal.



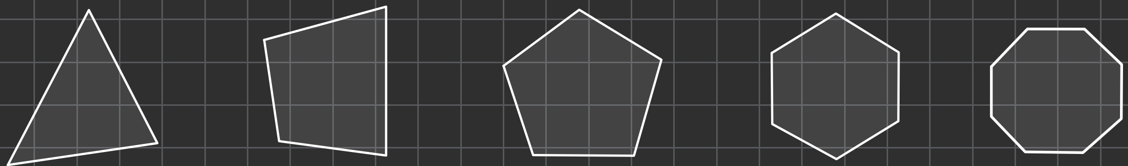
Corollary. Every polygon with $|V(P)| \geq 3$ has at least two ears.

Proof: Exercise

The number of triangulations of a fixed polygon P has much to do with the "shape".



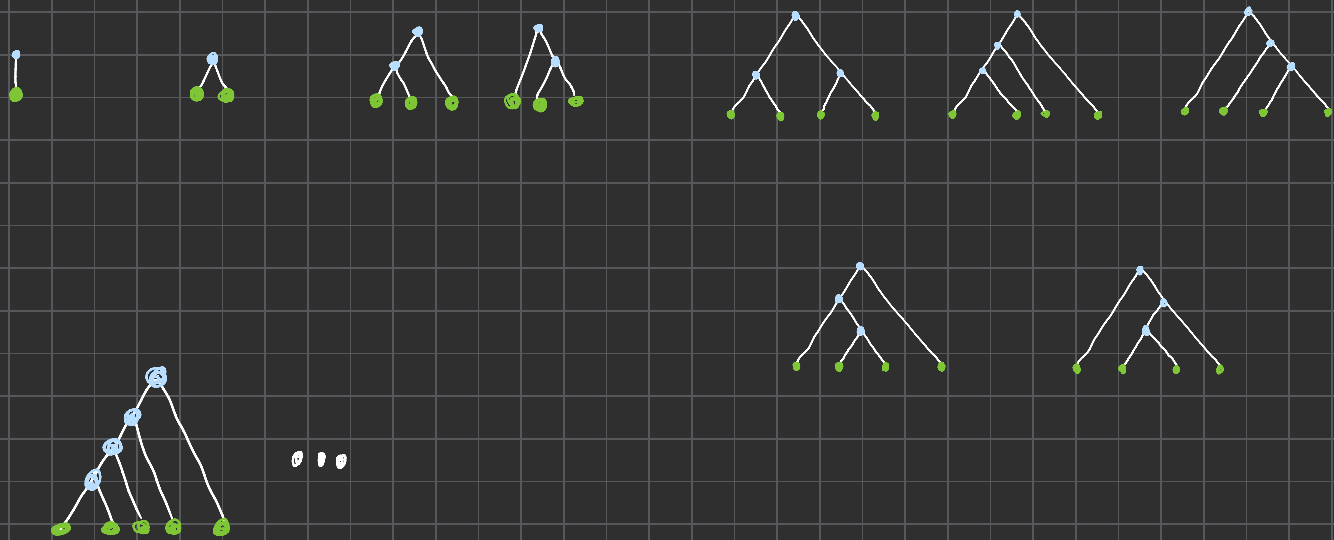
How many triangulations there are in a convex n-gon?



Binary Trees

A binary tree is a graph where each vertex has a maximum degree equal two.

The order of a binary tree is the number of vertices with degree 1 different to the root.



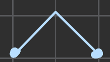
Dyck words

A word with alphabet consisting in only two letters, say $\{x,y\}$ is called Dyck-word if have the same number of x's and y's and in every "step" $\#x \geq \#y$

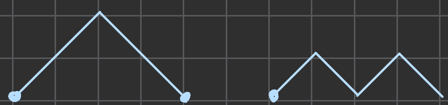
0	$\emptyset$	1
1	xy	1
2	xxyy, xyxy	2
3	xxxyyy, xxyxyy, xyxxyy, xyxyxy, xxyyxy	5
4	xxxxyyyy, ...	

A Dyck path is a lattice path in the plane that starts at the origin (0,0), consist of steps (1,1) (up) (1,-1) (down), stays on above the x-axis, and ends at the point (2n,0) for a non-negative integer n.

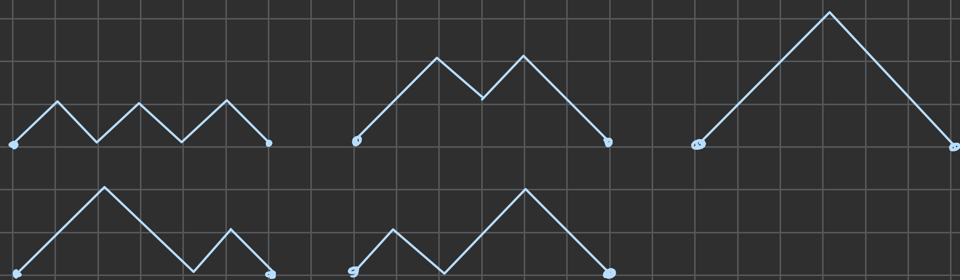
n=1



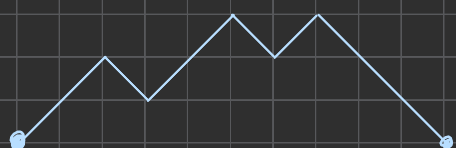
n=2



n=3

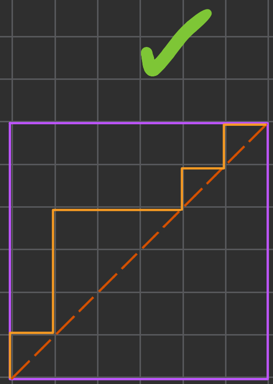


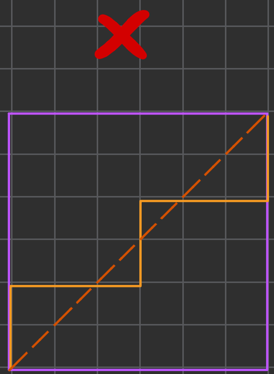
n=5



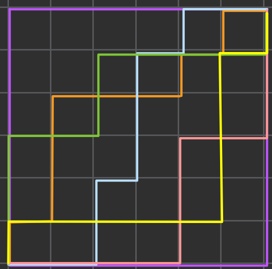
Lattice paths

How many northeast lattice paths from $(0,0)$ to (n,n) don't pass below the $x=y$ diagonal?





Let's count it!!

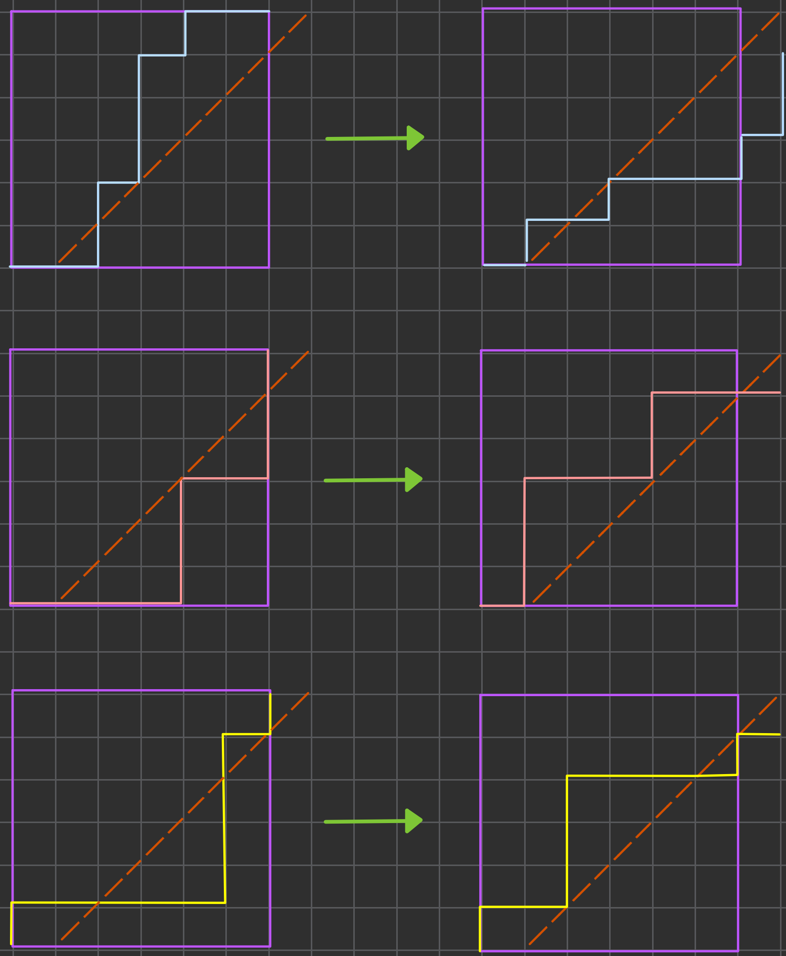


① First let's count all the possible paths.

There are ____ to many paths goin from (0,0) to (n,n)

Bad paths reflections

②



Observe that there is a bijection between every reflected bad path and the set of all possible paths going from (0,0) to (n+1, n-1).

There are ____ such paths.

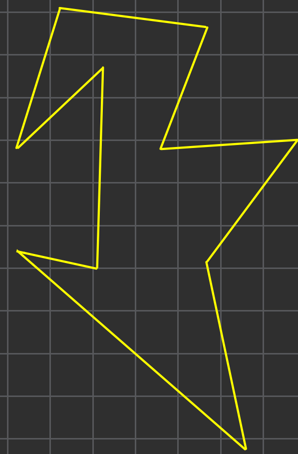
Finally by the inclusion-exclusion principle

$$C_n = \binom{2n}{n} - \binom{2n}{n+1} = \frac{1}{n+1}\binom{2n}{n}.$$

Then: ① A convex n-gon admit C_{n-2} triangulations.

② There are C_n B.trees / D. words / D. paths / R lattice paths of order n.

Art gallery problem. (by Klee)



Our gallery (in $\mathbb{R}^2$) is:

- A simple polygon P (no holes, no autointersections)

Our guards are:

- A set of points $S \subset P$

We said that our gallery is safe if

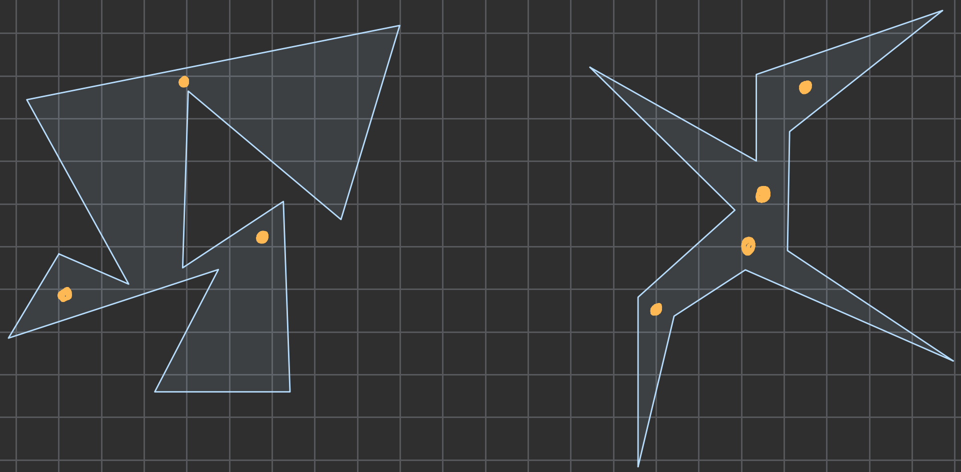
- Every point $p \in P$ can be "seen" by a point in S.

How many guards do we needs our gallery to be safe?

Can one guard keep safe the gallery?

If the guards are located in the corners (vertices) what is the small size of the set S?

We said that point x can see point y (or y is visible to x) iff the closed segment xy is nowhere exterior to the polygon P.



Two polygons of n=12 vertices: (a) requires 3 guards; (b) requires 4.

More formally:

Express as a function of n, the smallest number of guards that suffice to cover any polygon of n vertices.

Let $g(P)$ be the smallest number of guards needed to cover P.

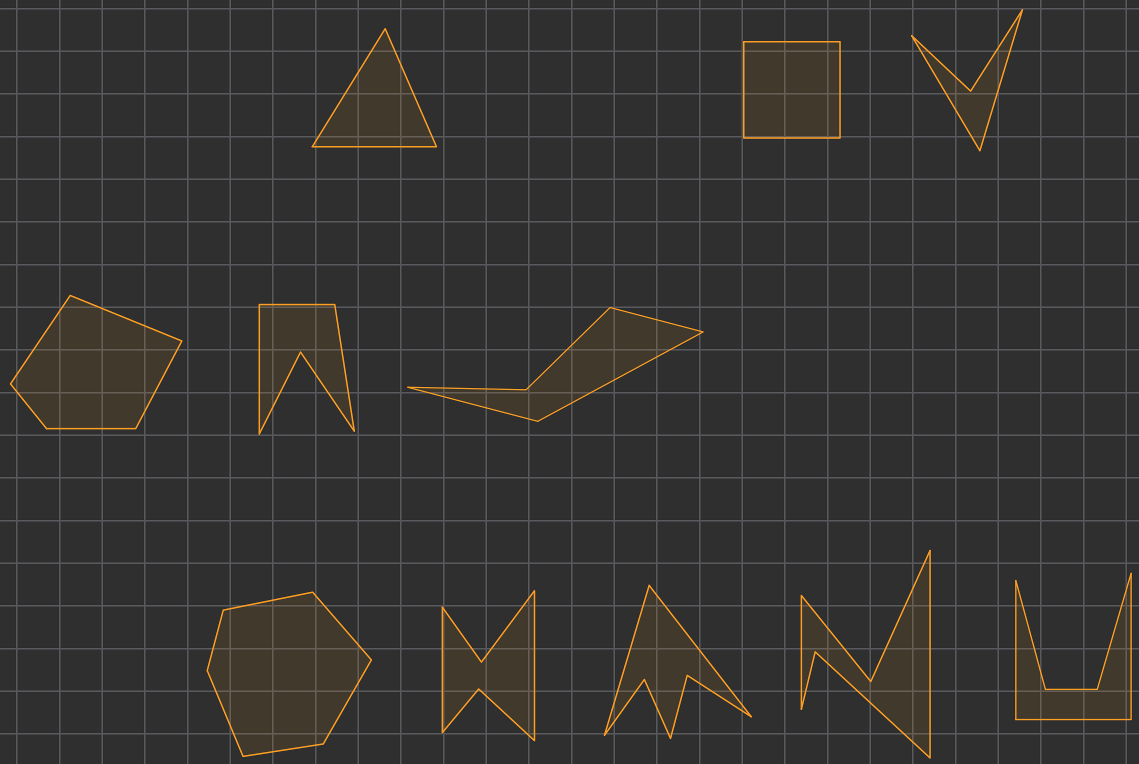
i.e., $$g(P) = \min_S |\{S : S \text{ covers } P\}|,$$

Let P_n be a polygon of n vertices, then we define

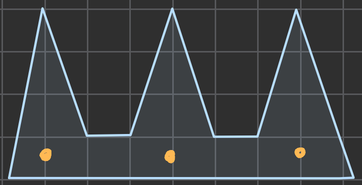
$$G(n) = \max_{P_n} g(P_n).$$

Then we are looking for $G(n)$.

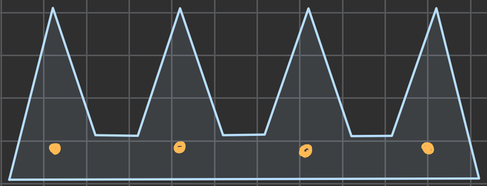
For a "small" n



We need at least $\lfloor n/3 \rfloor$



$n=9$



$n=12$

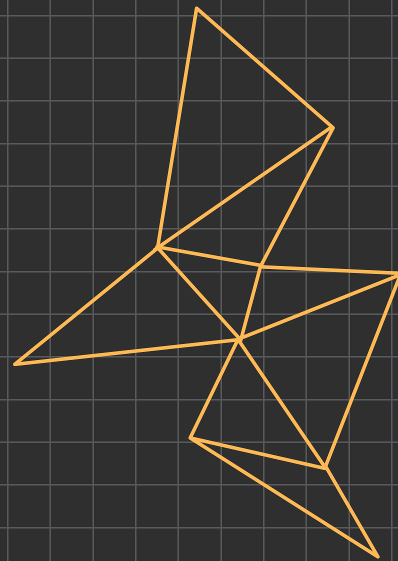
Chvátal construction

Then, is it true that $G(n) = \lfloor n/3 \rfloor$?

lemma: Every triangulation of a polygon is 3-colorable.

Proof: By induction on the number of vertices of P.

Base case: Consider the simple triangulation, a single triangle. Coloring each vertex with different colors there are no two adjacent with the same color.



Inductive hypothesis: Assume the lemma is valid for any triangulation of a polygon P with n vertices.

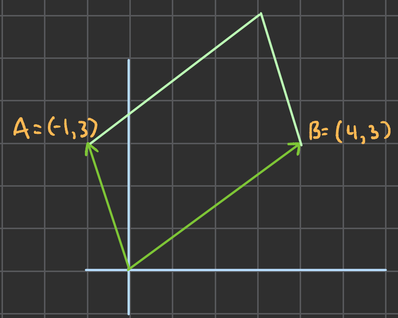
Inductive step: Now consider a polygon P with n+1 vertices. Choose a diagonal d that divides P into two smaller polygons P_1 and P_2. By inductive hypothesis these polygons can be 3-colored. Considering the colors assigned to the diagonal d in P_1 and perhaps after a possible permutation of the colors assigned to P_2, we obtain a 3-coloring of P.

Thm [Fisk 1978]: $G(n) = \lfloor \frac{n}{3} \rfloor$.

Proof: Chvátal construction give us $G(n) \geq \lfloor \frac{n}{3} \rfloor$.
By the lemma every triangulation T of a polygon P is 3-colorable. Since every point in P lies in a triangle $t \in T$ and every point in a triangle is visible for all its vertices, choosing one chromatic class we can see all the points of P.

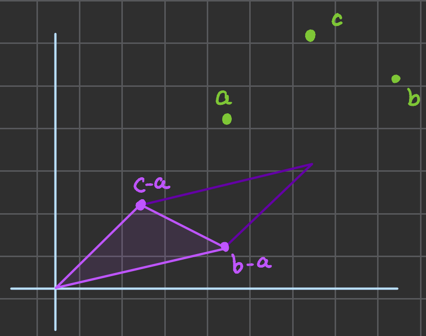
Area of a Triangle.

From linear algebra we know that if A and B are vectors, then the cross product $|A \times B|$ determine the area of the parallelogram with sides A and B.



$$|A \times B| = \left|\det\begin{pmatrix} -1 & 3 \\ 4 & 3 \end{pmatrix}\right| = |-3-12| = 15$$

Then for a,b,c points in $\mathbb{R}^2$ we have $\text{Area}\,\Delta abc = \frac{1}{2}|(b-a)\times(c-a)|$

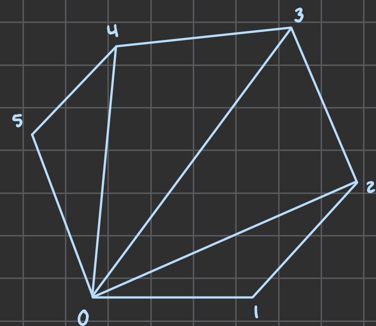


Lemma: Twice the area of a triangle $T=(a,b,c)$ is given by

$$2A(T) = \begin{vmatrix} a_0 & a_1 & 1 \\ b_0 & b_1 & 1 \\ c_0 & c_1 & 1 \end{vmatrix} = (b_0-a_0)(c_1-a_1)-(c_0-a_0)(b_1-a_1)$$

Area of a Polygon.

$$A(P) = A(v_0,v_1,v_2) + A(v_0,v_2,v_3) + \dots + A(v_0, v_{n-2}, v_{n-1}).$$



Area of a quadrilateral

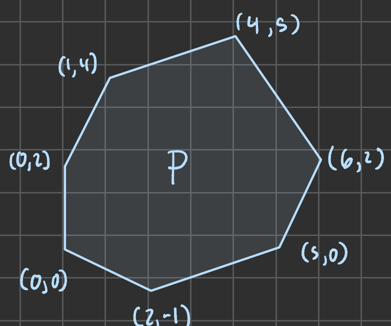
$$A(Q) = A(a,b,c) + A(a,c,d) = A(d,a,b) + A(d,b,c)$$

$$\Rightarrow \quad 2A(Q) = a_0b_1 - a_1b_0 + a_1c_0 - a_0c_1 + b_0c_1 - c_0b_1$$
$$+ a_0c_1 - a_1c_0 + a_1d_0 - a_0d_1 + c_0d_1 - d_0c_1$$

$$= a_0b_1 - a_1b_0 + b_0c_1 - c_0b_1 + a_1d_0 - a_0d_1 + c_0d_1 - d_0c_1$$
$$= a_0b_1 - a_1b_0 + b_0c_1 - b_1c_0 + c_0d_1 - c_1d_0 + d_0a_1 - d_1a_0$$

In general for a convex polygon P

$$2A(P) = \sum_{i=0}^{n-1} (x_i y_{i+1} - x_{i+1} y_i)$$



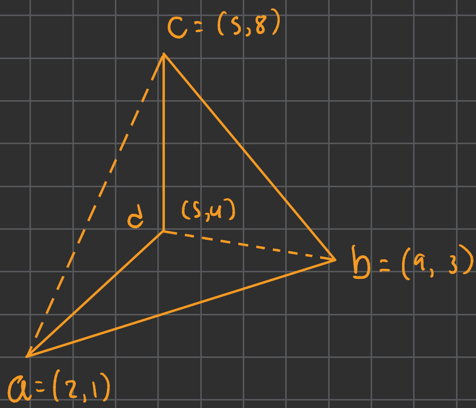
$$A(P)$$

$$\begin{vmatrix} 0 & 0 \\ 2 & -1 \\ 5 & 0 \\ 6 & 2 \\ 4 & 5 \\ 1 & 4 \\ 0 & 2 \end{vmatrix} = \frac{1}{2}\left|\left[0(-1)+2(0)+5(2)+6(5)+4(4)+1(2)\right] - \left[0(2)+(-1)(5)+0(6)+2(4)+5(1)+4(0)\right]\right|$$

$$= \frac{1}{2}\left|(10+30+16+2) - (-5+8+5)\right|$$

$$= 25$$

Area of a Nonconvex Quadrilateral.



$$\mathcal{A}(Q) = \mathcal{A}(a,b,c) + \mathcal{A}(a,c,d)$$

$$\mathcal{A}(Q) = \frac{1}{2}\begin{vmatrix} 2 & 1 & 1 \\ 9 & 3 & 1 \\ 5 & 8 & 1 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} 2 & 1 & 1 \\ 5 & 8 & 1 \\ 5, & 4 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[(6+72+5)-(15+16+9) + (16+5+20)-(40+8+5)\right]$$

$$= \frac{1}{2}\left[(83-40)+(41-53)\right]$$

$$= \frac{1}{2}\left[43-13\right]$$

$$= 15$$

Lemma 1.3.2. If $T = \Delta abc$ is a triangle with vertices oriented counterclokwise, and p is any point in the plane, then

$$A(T) = A(p,a,b) + A(p,b,c) + A(p,c,a),$$

Theorem [Area of Polygon]. Let a polygon (convex or nonconvex) P have vertices $v_0, \dots, v_{n-1}$ labeled counterclockwise, and let p be any point in the plane. Then

$$A(P) = A(p, v_{n-2}, v_{n-1}) + A(p, v_1, v_2) + A(p, v_2, v_3) + \dots + A(p, v_{n-2}, v_{n-1}) + A(p, v_{n-1}, v_0)$$

If $v_i = (x_i, y_i)$ $\Rightarrow$

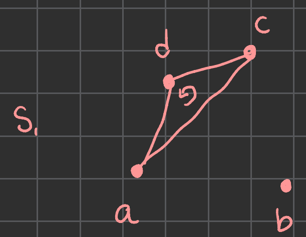
$$2A(P) = \sum_{i=0}^{n-1} (x_i y_{i+1} - y_i x_{i+1})$$

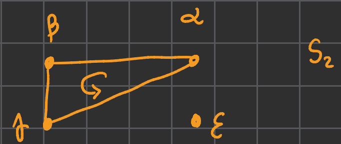
$$= \sum_{i=0}^{n-1} (x_i + x_{i+1})(y_{i+1} - y_i)$$

Ordertypes and chirotopes (for point sets in $\mathbb{R}^2$)

The order type of a set of points in the plane refers to the combinatorial information about the orientation of every triple of points in the set. Specifically, it describes whether each triple form a left turn a right turn or is collinear.

Two sets of points in $\mathbb{R}^2$ have the same order type if there is a one-one correspondence between their points that preserves the orientation of every triplet.



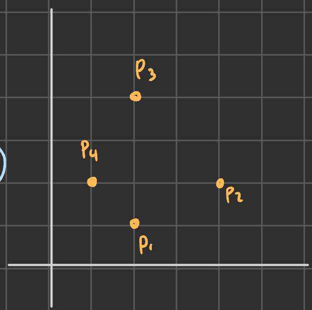


The function $f: S_1 \mapsto S_2$ where $f(a)=\alpha$, $f(b)=\beta$, $f(c)=\gamma$, $f(d)=\varepsilon$ satisfy that the orientation of every triplet T in S_1 is the same that the orientation of $f(T)$ en S_2.

For a set of points $P = \{p_1, \dots, p_n\}$ in $\mathbb{R}^d$, the chirotope χ is a function:

$$\chi : \{1, 2, \dots, n\}^{d+1} \longrightarrow (-, +, 0)$$

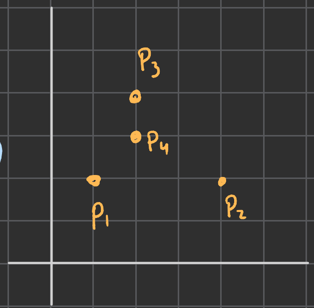
where $\chi(i_1, i_2, \dots, i_{d+1})$ represents the orientation of the $d+1$ points indexed by $\{i_1, \dots, i_{d+1}\}$.

① 

$$\chi(1,2,3) := \text{sign of} \begin{vmatrix} p_1 & 1 \\ p_2 & 1 \\ p_3 & 1 \end{vmatrix} = \text{sign of} \begin{vmatrix} 2 & 1 & 1 \\ 4 & 2 & 1 \\ 2 & 4 & 1 \end{vmatrix} = \text{sign of } 6 = +$$

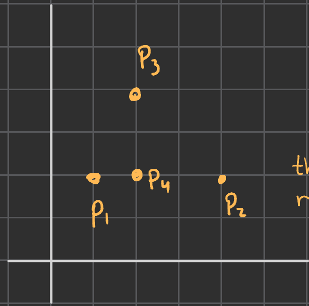
$\chi(1,2,4) = +$
$\chi(1,3,4) = +$
$\chi(2,3,4) = +$

then the chirotope is represented by $++++$.

② 

$\chi(1,2,3) = +$
$\chi(1,2,4) = +$
$\chi(1,3,4) = -$
$\chi(2,3,4) = +$

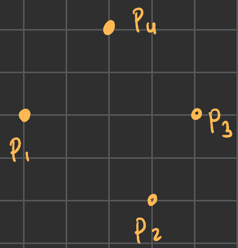
then the chirotope is represented by $++-+$

③ 

$\chi(1,2,3) = +$
$\chi(1,2,4) = 0$
$\chi(1,3,4) = -$
$\chi(2,3,4) = +$

then the chirotope is represented by $+0-+$

- The chirotope is antisymetric, meaning that swapping two indices the tuple the sign of the function changes.

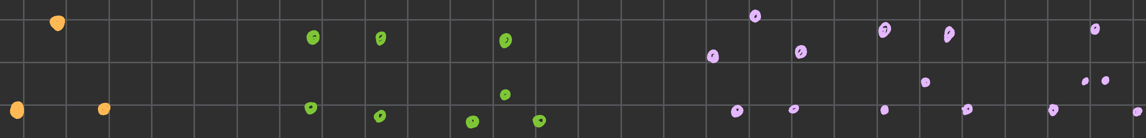


$\chi(P) = ++++$

$\chi(P) = ----$

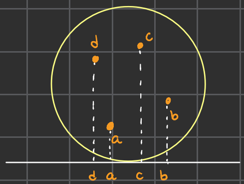
$\chi(P) = ++--$

How many order types there are?

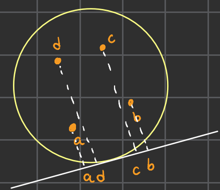


Number of points	Number of order types
3	1
4	2
5	3
6	16
7	135
8	3 315
9	158 817
10	14 309 547
11	2 334 512 907

Circular Sequences.



dacb



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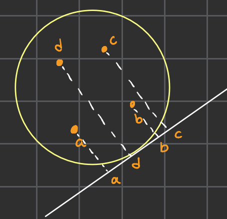
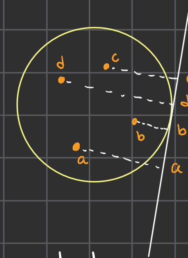
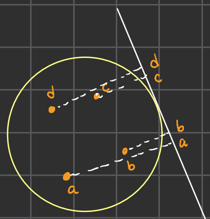


adbc



abdc



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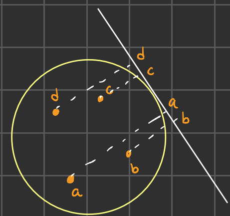
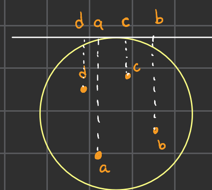


bacd



bcad

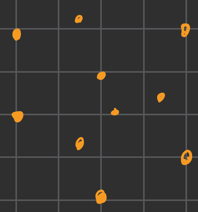
...

dacb
adcb
adbc
abdc
abcd
bacd
bcad
⋮

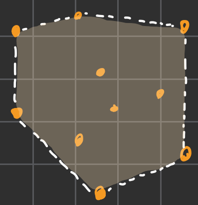
Convexity (I recommend to read Matousek's book).

A set $C \subseteq \mathbb{R}^d$ is convex if for every two points $x, y \in C$ the whole segment xy is also contained in C. In other words, for every $t \in [0,1]$, the point $tx + (1-t)y$ belongs to C.

The intersection of an arbitrary family of convex sets is obviously convex. So we can define the convex hull of a set $X \subseteq \mathbb{R}^d$, denoted by $\operatorname{conv}(X)$, as the intersection of all convex sets in $\mathbb{R}^d$ containing X.



X

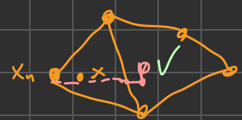


$\operatorname{conv}(X)$

Claim. A point x belongs to $\mathrm{conv}(X)$ iff there exist points $x_1, x_2, \dots, x_n \in X$ and nonegative real numbers $t_1, \dots, t_n$ with $\sum_{i=1}^{n} t_i = 1$ such that $x = \sum_{i=1}^{n} t_i x_i$.

Proof: $\Rightarrow$) By induction on the number of points.

- If $n=2$ this is by definition.
- Suppose this is valid for $n-1$ points
- Let x be a point in $\mathrm{conv}(X)$ where $|X|=n$.



$$p = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_{n-1} x_n \quad , \quad \sum \alpha_i = 1$$

$$x = tp + (1-t) x_n$$

$$= \alpha_1 t x_1 + \dots + \alpha_{n-1} t + (1-t) x_n$$

$$\sum_{i=1}^{n-1} \alpha_i t + (1-t) = t \sum_{i=1}^{n-1} \alpha_i + (1-t)$$

$$= t + (1-t)$$

$$= 1$$

$\Leftarrow$ The set of all convex conbinations contains X, and it is convex. ☑

Theorem (Carathéodbry's theorem). Let $X \subseteq \mathbb{R}^d$. Then each point of conv(X) is a convex conbination of at most d+1 points of X.

Proof: Let p be a point in the convex hull of X, then

$$p = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n$$

for some positive α_i's, s.t. $\sum_{i=1}^{n} \alpha_i = 1$.

If $n \leq d+1$ we are done.

Suppose then that $n > d+1$. Then the points $x_2 - x_1, x_3 - x_1, \ldots, x_n - x_1$ are linearly dependent. Let β_i $i = 2, \ldots, n$, be real numbers, not all zero, s.t

$$\sum_{i=2}^{n} \beta_i (x_i - x_1) = 0. \quad \leftarrow \text{Prove it}$$

So there are constants $\gamma_1, \ldots, \gamma_n$ not all zero, s.t

$$\sum_{i=1}^{n} \gamma_i x_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} \gamma_i = 0 \quad \leftarrow \text{Prove it}$$

Let $\mathcal{F}$ be a subset of positive scalars $\{i \in [n] : \gamma_i > 0\}$

$$a = \max_{i \in I} \frac{\alpha_i}{\gamma_i}$$

Then we have $p = \sum_{i=1}^{n} (\alpha_i - a\gamma_i) x_i$,

for at least one i this is no zero

in deed is gonna be $\alpha_i x_i$

$$\sum_{i=1}^{n} \alpha_i x_i - \sum_{i=1}^{n} a\gamma_i x_i$$

Observe that the sum of Coefficients is 1.

$\therefore$ We have a convex conv with les than $n > d+1$ points.

After repeating the above proces several times
we can express p as conv convination of at most $d+1$ pts

$\blacksquare$

Thm (Radon's lemma). Let A be a set of $d+2$ points in $\mathbb{R}^d$. Then there exist two disjoint subsets $A_1, A_2 \subset A$ s.t. $\operatorname{conv}(A) \cap \operatorname{conv}(B) \neq \emptyset$,

Proof: Let $A = \{a_1, \dots, a_{d+2}\}$, These points are affinely dependent. Then exists numbers $\alpha_1, \dots, \alpha_{d+2}$ not all of them 0 s.t.

$$\sum \alpha_i a_i = 0 \quad \text{and} \quad \sum \alpha_i = 0.$$

Set $P = \{i : \alpha_i > 0\}$, $N = \{i : \alpha_i < 0\}$

Let us put $A_1 = \{a_i : i \in P\}$ $A_2 = \{a_i : i \in N\}$. We are goin to exhibit a point x in the intersection of the conv hull of these sets.

Put $S = \sum_{i \in P} \alpha_i$ we have $S = -\sum_{i \in N} \alpha_i$

define $$x = \sum_{i \in P} \frac{\alpha_i}{S} a_i$$

Since $\sum_{i=1}^{d+2} \alpha_i a_i = 0 = \sum_{i \in P} \alpha_i a_i + \sum_{i \in N} \alpha_i a_i$ we also have $x = \sum_{i \in N} \frac{-\alpha_i}{S} a_i$ ∎

Thm (Helly). For a finite collection of convex sets $C_1, \dots, C_n \subset \mathbb{R}^d$, where $n > d$, if the intersection of every $d+1$ of these sets is nonempty, then

$$\bigcap_{i=1}^{n} C_i \neq \emptyset.$$

Proof: By induction on n. Since $n > d$ by hypothesis we have a base case. However we are going to show the case $n = d+2$, which will later be used in conjunction with the inductive hypothesis to prove the inductive step.

Choose a common point a_i of all sets C_j where $j \neq i$. i.e., $a_i \in \bigcap_{j \neq i} C_j$. Let $A = \{a_1, a_2, \dots, a_{d+2}\}$.

By Radon's Thm, there is a nontrivial, disjoint partition A_1, A_2 of A s.t $\mathrm{conv}(A_1) \cap \mathrm{conv}(A_2)$ intersect at some point x.

Also, observe that $\forall i \in [d+2]$, the only point that is not in C_i but is in A is a_i. Note that since $a_i \in A$, and $A = A_1 \cup A_2$, we can assume without loss of generality that $a_i \in A_1$. This means that $a_i \notin A_2$ so $A_2 \subset C_i$.
Since C_i is convex, it has to contain the convex hull of A_2 and in particular the point x.
Hence, x is common to all the C_i's, i.e., $x \in \bigcap_{i=1}^{d+2} C_i$.

A similar argument proves the cases $n > d+2$. $\boxtimes$

Thm (Helly). For a finite collection of convex sets $C_1, \dots, C_n \subset \mathbb{R}^d$, where $n > d$, if the intersection of every $d+1$ of these sets is nonempty, then

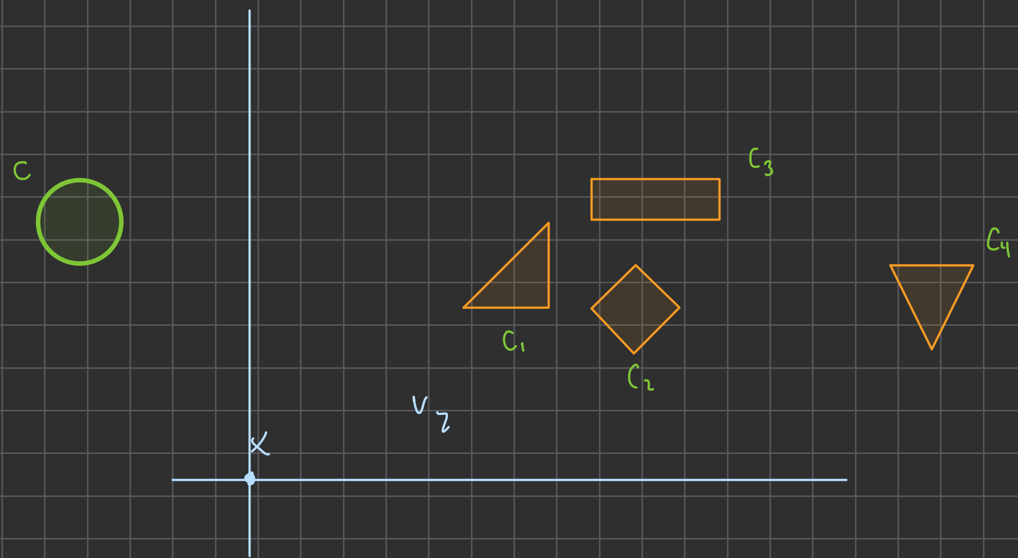
$$\bigcap_{i=1}^{n} C_i \neq \emptyset.$$

Exercises:

① Let $\mathcal{I}$ be a finite family of parallel line segments in $\mathbb{R}^2$, each three of which admit a common transversal. Then there is a common transversal to all members of $\mathcal{I}$.

Proof: We may suppose $\mathcal{I}$ consist in at least 3 members and all of them are parallel to the Y-axis.
For each segment $S \in \mathcal{I}$ Let $C_S = \{(a,b) \in \mathbb{R}^2 : S \cap l_{a,b}$ where $l_{a,b} := y = ax+b\}$.
Each C_S is convex and each 3 have nonempty intersection then by H.T. there is a point $(a_0,b_0) \in \bigcap_{S \in \mathcal{I}} C_S$.
The line $y = a_0 x + b_0$ is a transversal common to all members of $\mathcal{I}$.

② Consider a family of convex sets $\mathcal{F}=\{C_1,\dots,C_n\}$ in $\mathbb{R}^d$, and let C be a convex set in $\mathbb{R}^d$. If for every $d+2$ elemets of $\mathcal{F}$ there is a translation of C that intersect them, exist a translation of C that intersect all the convex sets in $\mathcal{F}$.



Proof: Let $C_i' = \{x\in\mathbb{R}^d : (x+C)\cap C_i \neq \emptyset\}$. Then each set C_i' is convex and each $d+1$ of these sets C_i''s have a common point. By H.T there exists a point x' in $\bigcap_{i=1}^{n} C_i'$, and $(x'+C)\cap C_i \neq \emptyset$ $\forall C_i\in\mathcal{F}$. $\boxtimes$